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COMMENT

Amplitude ratio of the second moments of the cluster size distribution on both sides of the percolation threshold

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Abstract. We propose in this comment an approximate form for the cluster size distribution function n_s. By means of this function, we can obtain not only the correct critical exponents of all moments of this distribution function, but the ratios of selected amplitudes as well, for all dimensions for which two critical exponents such as β and γ are known. We calculate explicitly the ratio C_+/C_- of the second moments of n_s for dimensions $d=2$, 3, 4, 5 and $6-\varepsilon$. In particular, for $6-\varepsilon$, our result agrees to order ε^2 with previous theoretical calculations, while for $d=2$ we obtain the large value 194, in good agreement with earlier Monte Carlo results.

1. Introduction

Universality in percolation applies not only to the critical exponents but also to selected ratios of the prefactors of various critical quantities [1-3].

Stauffer [4-6] has suggested, for the number of clusters of size s ($\gg 1$) near the percolation threshold, a distribution function of the form $n_s \simeq Ks^{-\tau}f(z)$, with $z=A(p-p_c)s^{\sigma}$, where p is the probability for a site to be active. The proportionality constants K and A, as well as the percolation threshold p_c, depend on the specific lattice on which the percolation problem is mapped, but the exponents τ and σ, closely related to the more commonly used β and γ, are found to be universal for all lattices of the same Euclidean dimension d. The function $f(z)$ is also presumed to be universal; it is one of the purposes of this comment to suggest an explicit form for this function.

Let us first outline a few definitions. For the sake of brevity, we shall confine our discussion to the problem of site percolation, though our results are equally valid for the bond percolation case. In order to emphasise the analogy with the temperature parameter in a magnetic transition, we define the deviation from the critical point as $t=(p_c-p)/p_c$. We let $n_s(t)$ represent the average number per *active* site of clusters of size s. The probability of an active site belonging to a finite cluster is then given by the first moment of the distribution n_s:

$$M_1=\sum_s sn_s(t). \tag{1}$$

Evidently then, below the threshold ($t>0$) $M_1=1$, as every active site belongs to a finite cluster. On the other hand, above the threshold, we have a non-zero probability that an active site belongs to the infinite cluster:

$$P_\infty=1-M_1\simeq B|t|^{\beta} \qquad (|t|\ll 1) \tag{2}$$

where it is readily shown that $\beta=(\tau-2)/\sigma$ [4-6]. If we now consider the second moment M_2 of n_s, we find that this is the expectation value, or 'mean', of the size of the cluster to which the site belongs. For percolation this is the exact mathematical analogue of the magnetic susceptibility, which diverges with an exponent γ:

$$M_2=\sum_s s^2 n_s(t)=S(t)\simeq C_\pm|t|^{-\gamma} \qquad (|t|\ll 1) \tag{3}$$

where again it is readily shown that $\gamma=(3-\tau)/\sigma$ [4-6]. The notation $C_\pm$ (the subscript corresponding to the sign of t) emphasises the fact that although the exponent of the divergence of M_2 is the same on both sides of the percolation threshold, the prefactors are not necessarily identical. It is in fact the main purpose of this comment to present an evaluation of the ratio

$$R=C_+/C_-=\lim_{t\to 0} S(+|t|)/S(-|t|). \tag{4}$$

It appears useful to state immediately the result which we shall obtain: starting with the exponent β, we show that it is sufficient to calculate an intermediate variable λ, obtained from the implicit equation

$$0=\int_0^\infty (\lambda-2x)x^{-\beta}\exp(\lambda x-x^2)\,\mathrm{d}x. \tag{5}$$

Once λ has thus been determined, the ratio $R=C_+/C_-$ is immediately calculated by inserting λ and the known value of γ in the ratio of the two definite integrals:

$$R=C_+/C_-=\left(\int_0^\infty x^{\gamma-1}\exp(\lambda x-x^2)\,\mathrm{d}x\right)\left(\int_0^\infty x^{\gamma-1}\exp(-\lambda x-x^2)\,\mathrm{d}x\right)^{-1}. \tag{6}$$

We see in table 1 the comparison of the results of our method, using currently accepted values of β and γ [2, 3], with published values of the ratio C_+/C_- [2, 7]. These latter, most often obtained from Monte Carlo calculations, are currently considered to be accurate to 10-20% at best. To this order of accuracy, our values, calculated as above, are indeed in satisfactory agreement. We further show, in the last section of this comment, that the use of our method and ε expansions of β and γ for dimension $6-\varepsilon$ gives an evaluation of the ratio R which agrees to order ε^2 with current theoretical expansions [1].

Table 1. Comparison of present results with published values for R†.

d	β	γ	λ (calculated)	R (calculated)	R (experimental) [2, 7]	$f_{\max}$ (calculated)	$f_{\max}$ (experimental) [5, 10]
2	0.139	2.39	2.517	194	$\begin{cases}219\pm25\\194\pm4\end{cases}$	4.87	4.5-5.4
3	0.42	1.76	1.314	9.03	8-11	1.54	1.6
4	0.57	1.46	0.898	3.75	$\simeq 5$	1.22	?
5	0.68	1.26	0.636	2.33	$\simeq 4$	1.106	?

† Calculated values are from our equations. $f_{\max}$ is the maximum value of the function $f_s(t)=n_s(t)/n_s(0)$.

2. Derivation of a 'shifted Gaussian' form for $n_s(t)$

The exact expression for the average distribution of clusters of size s per active site is

$$n_s = p^{s-1} \sum_u g_s(u)(1-p)^u = p^{s-1} G_s(p) \tag{7}$$

where $g_s(u)$ is the number of possible configurations of clusters of size s and perimeter u ($g_s(u)$ is a purely geometric factor independent of p).

If we carry out a Taylor expansion of the sum $G_s(p)$ in powers of the variable $t(=(p_c-p)/p_c)$, we obtain to second order in t

$$G_s(p) \simeq A_s(1+B_s t + C_s t^2 + \ldots) \tag{8}$$

where $A_s = \Sigma_u g_s(u)(1-p_c)^u$, $A_s B_s = p_c \Sigma_u u g_s(u)(1-p_c)^{u-1}$ and $2A_s C_s = p_c^2 \Sigma_u u(u-1) g_s(u)(1-p_c)^{u-2}$. For $s \gg 1$, equation (7) thus becomes

$$n_s(t) \simeq A_s p_c^s (1-t)^s (1+B_s t + C_s t^2) \tag{9}$$

which in the limit $|t| \ll 1$, and using the identity $1+at+bt^2 \simeq \exp[at-(a^2/2-b)t^2]$, we can write in exponential form:

$$n_s(t) \simeq n_s(0)\exp(D_s t - E_s t^2) \tag{10}$$

with $n_s(0) = A_s p_c^s$, $D_s = (B_s - s)$, $E_s = (B_s^2 + s - 2C_s)/2$.

At this stage, we recall the two conditions of the scaling hypothesis: that (i) $n_s(0) \simeq Ks^{-\tau}$ and that (ii) $f_s(t) = n_s(t)/n_s(0)$ be a function of the reduced variable ts^σ only. We must perforce then write $D_s = K_1 s^\sigma$, $E_s = K_2 s^{2\sigma}$, yielding

$$n_s(t) \simeq Ks^{-\tau}\exp(K_1 t s^\sigma - K_2 t^2 s^{2\sigma}) \tag{11}$$

where the constants K, K_1 and K_2 are all positive from physical considerations.

The above equation (11) for $n_s(t)$, derived here for $|t| \ll 1$, was in fact first proposed by Leath [8] on the basis of 'experimental' data. Upon the further introduction in $n_s(t)$ of a change of scale $T = t\sqrt{K_2}$ and a new constant $\lambda = K_1/\sqrt{K_2}$, we thus find that the ratio $R = C_+/C_-$, equation (4), becomes

$$R = \lim_{T\to 0^+} \frac{\Sigma_s s^2 n_s(T)}{\Sigma_s s^2 n_s(-T)} = \frac{\Sigma_s s^{2-\tau}\exp(\lambda T s^\sigma - T^2 s^{2\sigma})}{\Sigma_s s^{2-\tau}\exp(-\lambda T s^\sigma - T^2 s^{2\sigma})}. \tag{12}$$

All we therefore need to perform the evaluation of R is the appropriate value for the constant λ.

We recall, however, from our introductory remarks, that the first moment M_1 of the distribution n_s must be constant for all $T>0$. In particular, this implies that its first derivative $\mathrm{d}M_1/\mathrm{d}T$, evaluated by means of our expression (11), should approach zero for $T \to 0^+$, i.e.

$$\begin{aligned}\lim_{T\to 0^+}(\mathrm{d}M_1/\mathrm{d}T) &= \lim_{T\to 0^+}\sum_s \mathrm{d}/\mathrm{d}T(sn_s) \simeq \lim_{T\to 0^+}\int_1^\infty (\lambda s^\sigma - 2Ts^{2\sigma})s^{1-\tau}\exp(\lambda T s^\sigma - T^2 s^{2\sigma})\,\mathrm{d}s \\ &= \lim_{T\to 0^+}(T^{\beta-1}/\sigma)\int_T^\infty (\lambda - 2x)x^{-\beta}\exp(\lambda x - x^2)\,\mathrm{d}x \end{aligned} \tag{13}$$

where in the second integral expression we have used the substitutions $x = Ts^\sigma$ and $\beta = (\tau-2)/\sigma$. It is quite evident, therefore, that for $\beta < 1$ (i.e. not for the Bethe lattice) the divergence of the derivative can only be avoided by the vanishing of the coefficient integral (previously stated as equation (5))

$$a(\lambda, \beta) = 0 = \int_0^\infty (\lambda - 2x)x^{-\beta} \exp(\lambda x - x^2)\, dx.$$

By inspection this can only occur, for a given value of β, at a unique value of λ. (Reatto [9] seems to have been the first to have used such a relation in cluster models.) Figure 1 displays the solution of the implicit equation (5), which can be solved by various numerical methods.

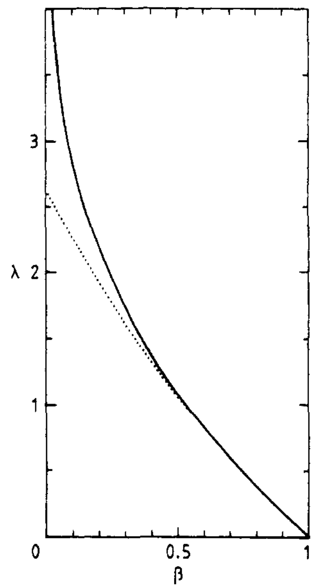

Figure 1. Dependence of λ on the critical exponent β, as obtained from equation (5). The dotted curve represents the approximate dependence given by series expansion (16) in $b = 1 - \beta$.

In exactly the same way as in equation (13), by transforming the sums in equation (12) to integrals and again making the substitutions $x = Ts^\sigma$ and $\gamma = (3 - \tau)/\sigma$, we immediately obtain equation (6) for the ratio R.

As previously indicated, the numerical results are displayed in table 1. We have added to this table the theoretical value of the maximum of the function $f_s(t) = n_s(t)/n_s(0)$, for which our model gives $\exp(\lambda^2/4)$, and which agrees remarkably well with previously known experimental values [5, 6, 10].

3. Calculation for dimension $6 - \varepsilon$

As the Euclidean dimension approaches the critical value $d = 6$, the exponent β approaches 1 from below while γ approaches 1 from above. We may therefore write $\beta = 1 - b$ and $\gamma = 1 + c$, with b and c as positive first-order corrections.

Integrating by parts, we can restate equation (5) in the form

$$a(\lambda, \beta) = 0 = \int_0^\infty (e^{\lambda x} e^{-x^2} - 1) x^{-\beta-1} \, dx \tag{14}$$

which, after replacing $e^{\lambda x}$ by its power series expansion, can be rewritten as an infinite sum of gamma functions

$$0 = \sum_{m=0}^{\infty} \lambda^m \{\Gamma[(b+m-1)/2]\}/m!. \tag{15}$$

Taking the lowest order terms of (15) in λ and b, we finally obtain the expression

$$\lambda = b\sqrt{\pi}\{1 + b(1 - \ln 2) + b^2[1 - \pi/4 + \pi^2/24 - \ln 2 + (\ln 2)^2/2] + \ldots\}. \tag{16}$$

In a similar way, it can be shown that

$$R = 1 + 2H\lambda + 2H^2\lambda^2 + \ldots \tag{17}$$

where

$$H = 1/\sqrt{\pi}\{1 + c \ln 2 + c^2[-\pi^2/24 + (\ln 2)^2/2] + \ldots\}.$$

Combining then the results of equations (16) and (17), we obtain to second order

$$R = 1 + 2b + b^2(4 - 2 \ln 2) + bc(2 \ln 2) + \ldots. \tag{18}$$

For dimension $d = 6 - \varepsilon$, the expansions for β and γ are [11]

$$\beta = 1 - \varepsilon/7 - \varepsilon^2 61/7^3 3^2 2^2 + \ldots$$

$$\gamma = 1 + \varepsilon/7 + \varepsilon^2 565/7^3 3^2 2^2 + \ldots$$

whence it follows that, to order ε^2,

$$R = 1 + 2\varepsilon/7 + \varepsilon^2 565/7^3 3^2 2 + \ldots \tag{19}$$

which is precisely Aharony's result [1].

4. Conclusions

The ideas of this comment represent a simple rederivation of results already well known in percolation theory. The 'shifted Gaussian' approximation for $n_s(t)$ appears to give excellent numerical and analytical results even though it might be in disagreement with the theorem of Kunz and Souillard [12] (see reference [5], p 32, for a thorough discussion of this question).

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